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## RIEMANN'S PROBLEM WITH CONTINUOUS COEFFICIENT

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**ABSTRACT:** A procedure for solving Riemann's boundary-value problem for the boundary between a multiply-connected region and its complement. Generalization of a result found previously by Gakhov by relaxing the conditions on the given functions. A slight change in one of the given functions does not appreciably change the result, so that a sequence of approximations is possible.

1°. Statement of the problem. Let  $C$  denote a contour consisting of  $m + 1$  simple closed contours  $C_0, C_1, \dots, C_m$  of the Lyapunov type that bound a connected region  $D^+$ . Its complement with respect to the plane consists of the union of  $m$  bounded simply-connected regions  $D_k^-$  (for  $k = 1, \dots, m$ ) and an infinite region  $D_0^-$ . For brevity, we shall refer to this complement as a region and shall denote it by  $D^-$ . Following the general practice, we denote by  $L_p(C)$  the class of functions that are  $p$ -summable on the contour  $C$ . /278\*

We formulate Riemann's problem as follows:

Find functions  $\Phi^\pm$  that are analytic\*\* in  $D^\pm$ , that have almost everywhere on the contour  $C$  limiting angular values  $\Phi^\pm(t)$  belonging to  $L_p(C)$  with  $p > 1$ , that satisfy the condition  $\Phi^-(\infty) = 0$ , and that satisfy the boundary condition

$$\Phi^+(t) = G(t) \Phi^-(t) + g(t), \quad (1)$$

where  $g(t) \in L_p(C)$  and  $G(t)$  is a continuous function that vanishes nowhere.

As usual, we call the integer  $\kappa = \sum_{k=0}^m \kappa_k$  where  $\kappa_k = \frac{1}{2\pi} [\arg G(t)]_{C_k}$ , the index of the problem.

We take as positive direction around a boundary of  $D^+$  that direction which puts  $D^+$  on one's left.

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\*Numbers in the margin indicate pagination in the foreign text.

\*\*We consider only functions that can be represented by a Cauchy integral.

Riemann's problem has been solved in closed form by F. D. Gakhov [1] for the case in which  $G(t)$  and  $g(t)$  satisfy a Hölder condition. B. V. Khvedelidze generalized this solution to the case of a multiply-connected region [2] when  $g$  is  $p$ -summable for  $p > 1$  [3].

In the present article, we show that Gakhov's results dealing with Riemann's problem remain valid when  $G(t)$  is assumed merely continuous.

2°. We know that every summable function can be represented in the form

$$\Phi(t) = \Phi^+(t) - \Phi^-(t), \quad (2)$$

where the  $\Phi^\pm(t)$  are almost everywhere limiting angular values of functions  $\Phi^\pm(z)$  that are analytic in  $D^\pm$ . The representation (2) is unique if we assume that  $\Phi^\pm(t) \in L_\delta(C)$  (with  $\delta > 0$ ) and  $\Phi^-(\infty) = 0$ .

The functions  $\Phi^\pm(t)$  are given by Sokhotskiy's formulas

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$$\begin{aligned} \Phi^+(t) &= \frac{1}{2} \Phi(t) + \frac{1}{2\pi i} \int_C \frac{\Phi(\tau)}{\tau - t} d\tau, \\ \Phi^-(t) &= -\frac{1}{2} \Phi(t) + \frac{1}{2\pi i} \int_C \frac{\Phi(\tau)}{\tau - t} d\tau. \end{aligned} \quad (3)$$

The validity of formulas (3) for summable  $\Phi$  was shown by I. I. Privalov [4]. B. V. Khvedelidze [3] showed that membership of  $\Phi(t)$  in the class  $L_p(C)$ , for  $p > 1$ , implies that  $\Phi^+(t)$  and  $\Phi^-(t)$  belong to  $L_p(C)$  also and that the singular operator  $\frac{1}{\pi i} \int_C \frac{\Phi(\tau)}{\tau - t} d\tau$  is a bounded operator in the norm of the space  $L_p(C)$ , for  $p > 1$ ; that is,  $\left( \int_C \left| \frac{1}{\pi} \int_C \frac{\Phi(\tau)}{\tau - t} d\tau \right|^p ds \right)^{1/p} \leq M_p \left( \int_C |\Phi(t)|^p ds \right)^{1/p}$ , where  $M_p$  is a constant independent of  $\Phi$ . It follows from this that

$$\|\Phi^\pm\|_{L_p} \leq \frac{1}{2} \|\Phi\|_{L_p} + \frac{M_p}{2} \|\Phi\|_{L_p} = \frac{M_p + 1}{2} \|\Phi\|_{L_p},$$

where

$$\|\Phi\|_{L_p} = \left( \int_C |\Phi(t)|^p ds \right)^{1/p}.$$

This enables us to formulate Riemann's problem in the following form.

Find a function  $\Phi$  belonging to the class  $L_p(C)$  and satisfying the boundary condition (1), where  $\Phi^+$  and  $\Phi^-$  denote the operators defined by equations (3).

Our investigation is based on the following simple idea: If  $G = 1$ , Riemann's problem reduces to a saltus problem and unconditionally has a unique solution in the class  $L_p$ . It turns out that small deviations in the coefficient  $G$  from unity do not change the nature of the solution. This can be proved by the method of

successive approximations. We can shift to the general case choosing a sequence of functions satisfying a Hölder condition that converges to  $\ln G$ . An analogous device was used by S. G. Mikhlin [5] in his investigation of singular integral equations with Cauchy kernel.

3°. Let us look at the case of Riemann's problem when  $G(t)$  is measurable and satisfies the condition

$$|G(t) - 1| \leq q < \frac{2}{1 + M_p}, \quad g(t) \in L_p(C) \quad (p > 1).$$

If we subtract  $\Phi^-$  from both sides of (1), we obtain

$$\Phi(t) = [G(t) - 1] \Phi^-(t) + g(t). \quad (4)$$

Note that

$$\begin{aligned} \|[G(t) - 1] \Phi^-\|_{L_p} &= \left( \int_C |G(t) - 1|^p |\Phi^-|^p ds \right)^{1/p} \leq \\ &\leq q \left( \int_C |\Phi^-|^p ds \right)^{1/p} = q \frac{1 + M_p}{2} \|\Phi^-\|_{L_p}. \end{aligned}$$

Applying the principle of contraction mappings, we see that the problem (4) (and consequently (1)) necessarily has a unique solution.

4°. The case  $\kappa_0 = \kappa_1 = \dots = \kappa_m = 0$ . In this case,  $\ln G$  is a continuous function. Let us approximate it with a function  $f$  that satisfies a Hölder condition and that satisfies the inequality

$$|e^{(\ln G - f)} - 1| \leq q < \frac{2}{1 + M_p}. \quad (5)$$

Furthermore, by using Gakhov's method, we can represent the function  $G_1(t) = e^f$  in the form of a ratio  $G_1(t) = X^+(t)/X^-(t)$ , where the  $X^\pm$  are functions that are analytic in the regions  $D^\pm + C$ , and nonzero everywhere.

Let us introduce new functions  $\Phi_1^\pm = \Phi^\pm [X]^{-1}$ , in terms of which the problem (1) can be written as follows:

$$\Phi_1^+ = \frac{G}{G_1} \Phi_1^- + \frac{g}{X^+}. \quad (6)$$

By virtue of condition (5), the coefficient  $GG_1^{-1}$  satisfies the requirements of section 3°. Therefore, the problem (6) and, hence, the problem (1) in the case  $\kappa_0 = \kappa_1 = \dots = \kappa_m = 0$  necessarily has a unique solution.

5°. Let us now investigate the most general case.

a)  $\kappa = 0$ . Introducing the new unknown functions

$$\Phi_{*}^{+}(z) = \prod_{k=1}^m (z - z_k)^{\kappa_k} \Phi^{+}(z), \quad \Phi_{*}^{-}(z) = \Phi^{-}(z) \quad (z_k \in D_k^{-}),$$

we arrive at the problem

$$\Phi_{*}^{+} = G^{*} \Phi_{*}^{-} + g^{*}, \quad (7)$$

where

$$G^{*}(t) = \prod_{k=1}^m (t - z_k)^{\kappa_k} G(t), \quad g^{*}(t) = \prod_{k=1}^m (t - z_k)^{\kappa_k} g(t).$$

We note that

$$\kappa_k^{*} = \frac{1}{2\pi} [\arg G^{*}]_{c_k} = 0, \quad k = 1, \dots, m; \quad \kappa_0^{*} = \frac{1}{2\pi} [\arg G^{*}]_{c_0} = \kappa = 0.$$

Therefore, on the basis of the results of section 4°, we conclude that the problem (6), and hence problem (1) in the case  $\kappa = 0$  have a unique solution.

b)  $\kappa > 0$ . Let us write the problem (1) in the following form:

$$\begin{aligned} & (t - z_0)^{-\kappa} \Phi^{+}(t) - (t - z_0)^{-\kappa} P_{\kappa-1}(t) = \\ & = (t - z_0)^{-\kappa} G(t) \Phi^{-}(t) + (t - z_0)^{-\kappa} g(t) - (t - z_0)^{-\kappa} P_{\kappa-1}(t), \end{aligned}$$

where  $z_0 \in D^{+}$ ;  $P_{\kappa-1}$  is a polynomial of degree not exceeding  $\kappa - 1$ , chosen in such a way that the function  $(z - z_0)^{-\kappa} [\Phi^{+}(z) - P_{\kappa-1}(z)]$  does not have a pole at the point  $z_0$ . We note that  $\text{Ind } (t - z_0)^{-\kappa} G = 0$ . We denote by  $R^{\pm}$  an operator solving Riemann's problem with coefficient  $(t - z_0)^{-\kappa} G$ . Then we have

$$\begin{aligned} (t - z_0)^{-\kappa} [\Phi^{+}(t) - P_{\kappa-1}(t)] &= R^{+}[(g(t) - P_{\kappa-1}(t))(t - z_0)^{-\kappa}], \\ \Phi^{-}(t) &= R^{-}[(g(t) - P_{\kappa-1}(t))(t - z_0)^{-\kappa}]. \end{aligned} \quad (8)$$

We find  $\Phi^{+}$  from the first equation:

$$\Phi^{+}(t) = P_{\kappa-1}(t) + (t - z_0)^{\kappa} R^{+}[(g(t) - P_{\kappa-1}(t))(t - z_0)^{-\kappa}]. \quad (9)$$

We have shown that the solution of the problem (1) must be of the form (8) or (9). However, one can easily see that, for an arbitrary polynomial  $P_{\kappa-1}$ , formulas (8) and (9) yield the general solution of the problem. /281

c)  $\kappa < 0$ . We write the problem (1) in a different form:

$$\Phi^{+}(t) (t - z_0)^{-\kappa} = G(t) (t - z_0)^{-\kappa} \Phi^{-}(t) + g(t) (t - z_0)^{-\kappa}.$$

Obviously,  $\text{Ind } (t - z_0)^{-\kappa} G(t) = 0$  and  $(t - z_0)^{-\kappa} \Phi^{+}(t)$  is analytic in  $D^{+}$ . If we again denote by  $R^{\pm}$  an operator that solves Riemann's problem with coefficient

$(t - z_0)^{-\kappa} G$ , we obtain

$$\Phi^+(z) = (z - z_0)^{\kappa} R^+ [(t - z_0)^{-\kappa} g(t)], \quad (10)$$

$$\Phi^-(z) = R^- [(t - z_0)^{-\kappa} g(t)]. \quad (11)$$

The function defined by the expression (10) has, in general, a pole of order  $|\kappa|$  at the point  $z_0$ . Therefore, for the problem posed to have a solution,  $|\kappa|$  solvability conditions must be satisfied:

$$\int_C (t - z_0)^{-\kappa} R^+ [g(t) (t - z_0)^{-\kappa}] dt = 0. \quad (12)$$

When (12) is satisfied, the problem has a unique solution.

Summarizing the results of section 5°, we have:

**Theorem.** a) In the case  $\kappa = 0$ , Riemann's boundary problem has a unique solution.

b) In the case  $\kappa > 0$ , the problem is always solvable, and its general solution has  $\kappa$  linearly independent components.

c) In the case  $\kappa < 0$ , the problem has a solution only when  $|\kappa|$  solvability conditions  $S_k g = 0$  are satisfied, where the  $S_k$  are linearly independent functionals. When these conditions are satisfied, the problem has a unique solution.

In conclusion, I wish to express my deep gratitude to F. D. Gakhov, who supervised the work, and to V. V. Ivanov for useful criticism of it.

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